SUPPORT THEOREMS FOR DEGENERATE STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS AND APPLICATIONS*

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Abstract. In the paper, we are concerned with degenerate stochastic differential equations with jumps. Firstly, we establish two support theorems for the solutions of the degenerate stochastic equations, under different (sufficient) conditions. Secondly, we apply one of our support theorems to a class of degenerate stochastic evolution equations (i.e., infinite-dimensional stochastic differential equations) with jumps to get a characterisation of path-independence for the densities of their Girsanov transformations.

1. Introduction

Support theorems for stochastic differential equations are referred to as the supports for the laws (or the distributions) of their solutions, or equivalently, that the state spaces for the solutions are characterised under certain topologies. Support theorem for diffusion processes governed by stochastic differential equations was initiated in the two seminar papers Stroock-Varadhan [16, 17]. Since then, there have been a lot of results on this topic. Let us recall some works here. In [1], Aida extended the celebrated Stroock-Varadhan support theorems for finite-dimensaion diffusion processes to the infinite-dimensional case. In [4], Bally-Millet-Sanz-Solé investigated supports for the solution distributions of solutions of parabolic stochastic partial differential equations in Hölder norm. Simon [15] studied a type of stochastic differential equations driven by (compensated or not) infinite Poisson measures and obtained support theorems for these equations. Furthermore, Fournier [7] considered a class of parabolic stochastic partial differential equations driven by space-time Gaussian white noises and independent Poisson measures, and proved that the supports of their solution distributions are characterized as the closures of sets of weak solutions for ordinary partial differential equations. There, the author required that the coefficients of their continuous diffusion terms are non-degenerate.

In the paper, we are concerned with a class of stochastic differential equations driven by Brownian motions and independent compensated Poisson measures. And we emphasize that the coefficient of the continuous diffusion term may be degenerate. Under certain suitable assumptions, two support theorems are shown. And then we make use of one

AMS Subject Classification(2010): 60H10.

Keywords: Support theorems, degenerate stochastic differential equations with jumps, path-independence, infinite-dimensional integro-differential equations.

*This work was partly supported by NSF of China (No. 11371352, 11671083).

support theorem to a problem on degenerate stochastic evolution equations with jumps on Hilbert spaces. By some deduction, path-independence for the densities of their Girsanov transformations is characterised.

It is worthwhile to mentioning our previous results. In [13, 14], we showed that these densities of Girsanov transformations for stochastic differential equations with jumps and stochastic evolution equations with jumps are path-independent, when the coefficients of their continuous diffusion terms are non-degenerate. And then the second named author and B. Wu [19] only mentioned that these densities of Girsanov transformations for degenerate stochastic differential equations are path-independent. Here in this paper, we not only permit that their continuous diffusion coefficients are degenerate, but also give some concrete conditions and detailed proof.

This rest of the paper is organized as follows. In Section 2, we prove two support theorems for stochastic differential equations with jumps under different (sufficient) conditions. Section 3 is devoted to applying a support theorem to a problem on degenerate stochastic evolution equations with jumps. By an extended Itô formula, we obtain path-independence for the density of their Girsanov transformation.

The following convention will be used throughout the paper: C with or without indices will denote different positive constants (depending on the indices) whose values may change from one place to another.

2. Support theorems

In the section, we will prove two support theorems for stochastic differential equations with jumps applied in the next section.

Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$ be a complete filtered probability space. Fix $T > 0$ and consider the following SDE with jumps on $\mathbb{R}^d$

$$\left\{\begin{array}{ll} \mathrm{d}Z_t = \xi(Z_t)\mathrm{d}t + \eta(Z_t)\mathrm{d}B_t + \int_{\mathbb{U}_0} \zeta(Z_{t-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), & t \in (0, T], \\ Z_0 = z_0 \in \mathbb{R}^d. \end{array}\right. \tag{1}$$

Here $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ is a finite dimensional normed space with its Borel σ-algebra $\mathscr{U}$. Let ν be a σ-finite measure defined on $(\mathbb{U}, \mathscr{U})$. We fix $\mathbb{U}_0 \in \mathscr{U}, \mathbb{U}_0 \subset \mathbb{U} - \{0\}$ with $\nu(\mathbb{U} \setminus \mathbb{U}_0) < \infty$ and $\int_{\mathbb{U}_0} \|u\|_{\mathbb{U}}^2 \nu(\mathrm{d}u) < \infty$. Moreover, we can construct an integer-valued $(\mathscr{F}_t)_{t\geqslant 0}$-Poisson random measure $N(\mathrm{d}t, \mathrm{d}u)$ on $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$ with the intensity $\mathrm{d}t\nu(\mathrm{d}u)$. Set

$$\tilde{N}(\mathrm{d}t, \mathrm{d}u) := N(\mathrm{d}t, \mathrm{d}u) - \mathrm{d}t\nu(\mathrm{d}u)$$

and then $\tilde{N}(\mathrm{d}t, \mathrm{d}u)$ is the compensated $(\mathscr{F}_t)_{t\geqslant 0}$-predictable martingale measure of $N(\mathrm{d}t, \mathrm{d}u)$. Here $\{B_t\}$ is a d-dimensional $(\mathscr{F}_t)_{t\geqslant 0}$-Brownian motion, which is independent of $N(\mathrm{d}t, \mathrm{d}u)$. The coefficients $\xi : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\eta : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ and $\zeta : \mathbb{R}^d \times \mathbb{U}_0 \mapsto \mathbb{R}^d$ are all Borel measurable.

Assumption 1.

(i) There exists a constant $L_1 > 0$ such that for any $z, z_1, z_2 \in \mathbb{R}^d$ and $u, u_1, u_2 \in \mathbb{U}_0$,

$$|\xi(z_1) - \xi(z_2)|^2 + \|\eta(z_1) - \eta(z_2)\|^2 \leqslant L_1|z_1 - z_2|^2,$$

where $\|\cdot\|$ stands for the Hilbert-Schmidt norm of a matrix, and

$$\begin{aligned} |\zeta(z_1, u) - \zeta(z_2, u)| &\leqslant L_1|z_1 - z_2|\|u\|_{\mathbb{U}}, \\ |\zeta(z, u_1) - \zeta(z, u_2)| &\leqslant L_1(1 + |z|)\|u_1 - u_2\|_{\mathbb{U}}. \end{aligned}$$

(ii) There exists a constant $L_2 > 0$ such that for any $z \in \mathbb{R}^d$

$$|\xi(z)|^2 + \|\eta(z)\|^2 + \int_{\mathbb{U}_0} |\zeta(z,u)|^2 \nu(\mathrm{d}u) \leqslant L_2(1+|z|^2).$$

Under **Assumption 1**, it is known that there exists a unique strong solution $\{Z_t\}_{t\in[0,T]}$ to Eq.(1) which is a Markov process with càdlàg paths, see, e.g., [3, Theorem 6.2.3 and Theorem 6.4.5]. Next, we define the support for a random variable and then we study the support of Z_t for $t \in [0,T]$.

Definition 2.1. *Let $\mathbb{Z}$ be a metric space with the metric ρ. The support of a $\mathbb{Z}$-valued random variable γ is defined to be*

$$supp(\gamma) := \{z \in \mathbb{Z} | (\mathbb{P} \circ \gamma^{-1})(B(z,r)) > 0, \text{ for all } r > 0\},$$

where $B(z,r) := \{y \in \mathbb{Z} | \rho(z,y) < r\}$.

Assumption 2.
For any $z \in \mathbb{R}^d$ and any open ball $B \subset \mathbb{R}^d$, there exists a point $u \in supp(\nu^{\mathbb{U}_0})$, the restriction of ν to $\mathbb{U}_0$, such that $\zeta(z,u) \in B$.

Now, we state and prove the first main result of this section.

Theorem 2.2. *Suppose that ξ, η, ζ satisfy* **Assumption 1** *and* **Assumption 2**. *Then $supp(Z_t) = \mathbb{R}^d$ for $t \in [0,T]$.*

Proof. By Definition 2.1, it is sufficient to prove that for any $t \in [0,T], a \in \mathbb{R}^d$ and $r > 0$

$$\mathbb{P}(Z_t \in B(a,r)) > 0.$$

And then we fix t, a, r and prove the above inequality with the help of two auxiliary processes.

Step 1. For any subset $U \subset \mathbb{U}_0$ with $U \in \mathscr{U}, \nu(U) < \infty$ and $L_1 \|u\|_{\mathbb{U}} < 1$ for $u \in U$, we introduce an auxiliary equation

$$Z_t^U = z_0 + \int_0^t \xi(Z_s^U)\mathrm{d}s + \int_0^t \int_U \zeta(Z_{s-}^U, u)\tilde{N}(\mathrm{d}s, \mathrm{d}u). \tag{2}$$

Under **Assumption 1**, it follows from [3, Theorem 6.2.3] that Eq(2) has a unique solution denoted as Z^U. Thus, Lemma 2.3 below admits us to obtain that for any $r > \varepsilon > 0$, there exists a $n \in \mathbb{N}$ such that

$$\sup_{0 \leqslant s \leqslant t_n} |Z_s - Z_s^U| < \varepsilon/2. \tag{3}$$

Fix this t_n and ε.

Step 2. Let $\{s_i\}$ be a positive sequence such that $s_i \uparrow \infty$, and $\{u_i\}$ be a sequence in the support of ν^U, the restriction of ν to U. And let $\mathbb{G}^U$ be the collection of the above sequences pair $\{s_i\}, \{u_i\}$. For any $g \in \mathbb{G}^U$, we introduce the second auxiliary equation

$$Z_t^{g,U} = z_0 + \int_0^t \left[\xi(Z_s^{g,U}) - \int_U \zeta(Z_s^{g,U}, u)\nu(\mathrm{d}u)\right] \mathrm{d}s + \sum_{i: s_i \leqslant t} \zeta(Z_{s_i-}^{g,U}, u_i). \tag{4}$$

Under **Assumption 1**, it holds that Eq.(4) has a unique solution denoted as $Z^{g,U}$. So, by **Assumption 2**, we know that for the open ball $B(a, r-\varepsilon)$, there exist $s_1 > 0, s_i > t_n, i = 2, 3, \dots$ and $u_1 \in supp(\nu^U)$ such that

$$|Z^{g,U}_{t_n} - a| < r - \varepsilon. \tag{5}$$

Fix $\{s_i\}, u_1$.

Step 3. We study the relationship between Z^U and $Z^{g,U}$. Set

$$\chi_t := \int_0^t \int_U u\tilde{N}(\mathrm{d}s, \mathrm{d}u), \quad \Delta\chi_t := \chi_t - \chi_{t-}, \quad D := \{t \in [0,\infty), \Delta\chi_t \in U\},$$

and then it follows from $\nu(U) < \infty$ that D is a discrete set in $[0,\infty)$ a.s.. Let $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ be the enumeration of all elements in D. Besides, we take any $u_2 \in supp(\nu^U)$. For any $\varepsilon' > 0$, set

$$\begin{aligned} A_1 &:= \{\omega \in \Omega : 0 < s_1 - \tau_1 < \varepsilon', \|u_1 - \Delta\chi_{\tau_1}\|_{\mathbb{U}} < \varepsilon'\}, \\ A_2 &:= \{\omega \in \Omega : 0 < s_2 - s_1 - (\tau_2 - \tau_1) < \varepsilon', \|u_2 - \Delta\chi_{\tau_2}\|_{\mathbb{U}} < \varepsilon'\}, \end{aligned}$$

and then it follows from independence of $N(\mathrm{d}t, \mathrm{d}u)$ that $\mathbb{P}(A_1 \cap A_2) > 0$. Thus, by Lemma 2.4 below it holds that for the above ε, there exists an $\varepsilon' > 0$ such that

$$\sup_{0\leqslant s\leqslant t_n} |Z^U_s - Z^{g,U}_s| < \varepsilon/2, \tag{6}$$

on $A_1 \cap A_2$.

Step 4. Combining (3) (5) with (6), we obtain that

$$|Z_s - a| \leqslant |Z_s - Z^U_s| + |Z^U_s - Z^{g,U}_s| + |Z^{g,U}_s - a| < \varepsilon/2 + \varepsilon/2 + r - \varepsilon = r, \quad s \in (0, t_n],$$

on $A_1 \cap A_2$. Thus $\mathbb{P}(|Z_s - a| < r) > 0$ for $s \in (0, t_n]$. If $t \leqslant t_n$, the proof is over; if $t > t_n$, by the Markov property, we still can obtain $\mathbb{P}(|Z_t - a| < r) > 0$. The proof is completed. □

Lemma 2.3. *Under* **Assumption 1**, *there exists a positive (nonrandom) sequence* $\{t_n\}$ *decreasing to* 0 *such that*

$$\lim_{n\to\infty} \sup_{0\leqslant s\leqslant t_n} |Z_s - Z^U_s| = 0, \quad a.s. \,\mathbb{P}.$$

Proof. Firstly, we compute $Z_t - Z^U_t$ for $t \in [0, T]$. By (1) and (2), it holds that

$$\begin{aligned} Z_t - Z^U_t &= \int_0^t \left(\xi(Z_s) - \xi(Z^U_s)\right) \mathrm{d}s + \int_0^t \eta(Z_s) \mathrm{d}B_s \\ &\quad + \int_0^t \int_U \left(\zeta(Z_{s-}, u) - \zeta(Z^U_{s-}, u)\right) \tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_0^t \int_{\mathbb{U}_0\setminus U} \zeta(Z_{s-}, u)\tilde{N}(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

And by the BDG inequality and the Höld inequality, one can have that

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leqslant s\leqslant t} |Z_s - Z^U_s|^2\right) &\leqslant 4t\mathbb{E}\int_0^t \left|\xi(Z_s) - \xi(Z^U_s)\right|^2 \mathrm{d}s + 4\mathbb{E}\int_0^t |\eta(Z_s)|^2 \mathrm{d}s \\ &\quad +4\mathbb{E}\int_0^t \int_U \left|\zeta(Z_{s-}, u) - \zeta(Z^U_{s-}, u)\right|^2 \nu(\mathrm{d}u)\mathrm{d}s \end{aligned}$$

$$+4\mathbb{E}\int_0^t\int_{\mathbb{U}_0\setminus U}|\zeta(Z_{s-},u)|^2\nu(\mathrm{d}u)\mathrm{d}s.$$

Moreover, by **Assumption 1**, we obtain that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right) &\leqslant 4L_1(t+1)\int_0^t\mathbb{E}\left(\sup_{0\leqslant s\leqslant r}|Z_s-Z_s^U|^2\right)\mathrm{d}r\\ &\quad +4\mathbb{E}\int_0^t L_2(1+|Z_s|^2)\mathrm{d}s. \end{aligned}\tag{7}$$

To estimate the last term in (7), we observe Eq.(1). By similar deduction to above, one can get that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s|^2\right) &\leqslant 4|z_0|^2+4t\mathbb{E}\int_0^t|\xi(Z_s)|^2\,\mathrm{d}s+4\mathbb{E}\int_0^t|\eta(Z_s)|^2\mathrm{d}s\\ &\quad +4\mathbb{E}\int_0^t\int_{\mathbb{U}_0}|\zeta(Z_{s-},u)|^2\nu(\mathrm{d}u)\mathrm{d}s,\end{aligned}$$

and furthermore by **Assumption 1**

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s|^2\right) &\leqslant 4|z_0|^2+4(t+1)L_2\int_0^t\mathbb{E}\left(1+|Z_s|^2\right)\mathrm{d}s\\ &\leqslant 4|z_0|^2+4(t+1)tL_2+4(t+1)L_2\int_0^t\mathbb{E}\left(\sup_{0\leqslant s\leqslant r}|Z_s|^2\right)\mathrm{d}r.\end{aligned}$$

Thus, the Gronwall inequality admits us to have that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s|^2\right)\leqslant C,\tag{8}$$

where the constant $C>0$ depends on z_0,T,L_2.

Next, combining (7) with (8), we get that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right) \leqslant 4L_2(C+1)t+4L_1(T+1)\int_0^t\mathbb{E}\left(\sup_{0\leqslant s\leqslant r}|Z_s-Z_s^U|^2\right)\mathrm{d}r.$$

Based on the Gronwall inequality, it holds that

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}|Z_s-Z_s^U|^2\right)\leqslant C(e^{Ct}-1),$$

where the constant $C>0$ depends on z_0,T,L_1,L_2. Set $t_n:=C^{-1}\ln(1+2^{-n})$, for $n=1,2,\cdots$, and then

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t_n}|Z_s-Z_s^U|^2\right)\leqslant C2^{-n}.$$

Thus, there exists a subsequence still denoted as $\{t_n\}$ such that

$$\lim_{n\to\infty}\sup_{0\leqslant s\leqslant t_n}|Z_s-Z_s^U|=0,\quad a.s.\mathbb{P}.$$

The proof is completed. $\square$

Lemma 2.4. *Under* **Assumption 1***, for the above ε, there eixsts an $\varepsilon' > 0$ such that*

$$\sup_{0\leqslant s\leqslant t_n} |Z_s^U - Z_s^{g,U}| < \varepsilon/2,$$

on $A_1 \cap A_2$.

Proof. By (2) and (4), it holds that for $0 \leqslant t \leqslant t_n$

$$\begin{aligned} Z_t^U - Z_t^{g,U} &= \int_0^t \left[\xi(Z_s^U) - \xi(Z_s^{g,U})\right] \mathrm{d}s + \int_0^t \int_U \left[\zeta(Z_s^U, u) - \zeta(Z_s^{g,U}, u)\right] \nu(\mathrm{d}u)\mathrm{d}s \\ &\quad + \zeta(Z_{\tau_1-}^U, \Delta\chi_{\tau_1}) - \zeta(Z_{s_1-}^{g,U}, u_1), \end{aligned}$$

and

$$\begin{aligned} |Z_t^U - Z_t^{g,U}| &\leqslant \int_0^t \left|\xi(Z_s^U) - \xi(Z_s^{g,U})\right| \mathrm{d}s + \int_0^t \int_U \left|\zeta(Z_s^U, u) - \zeta(Z_s^{g,U}, u)\right| \nu(\mathrm{d}u)\mathrm{d}s \\ &\quad + |\zeta(Z_{\tau_1-}^U, \Delta\chi_{\tau_1}) - \zeta(Z_{s_1-}^U, \Delta\chi_{\tau_1})| + |\zeta(Z_{s_1-}^U, \Delta\chi_{\tau_1}) - \zeta(Z_{s_1-}^U, u_1)| \\ &\quad + |\zeta(Z_{s_1-}^U, u_1) - \zeta(Z_{s_1-}^{g,U}, u_1)|. \end{aligned}$$

So, **Assumption 1** admits us to obtain that

$$\begin{aligned} \sup_{0\leqslant s\leqslant t_n} |Z_s^U - Z_s^{g,U}| &\leqslant L_1 \int_0^{t_n} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}| \mathrm{d}r \\ &\quad + L_1 \left(\int_U \|u\|_{\mathbb{U}} \nu(\mathrm{d}u)\right) \int_0^{t_n} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}| \mathrm{d}r \\ &\quad + L_1 \|\Delta\chi_{\tau_1}\|_{\mathbb{U}} |Z_{\tau_1-}^U - Z_{s_1-}^U| + L_1(1 + |Z_{s_1-}^U|)\|\Delta\chi_{\tau_1} - u_1\|_{\mathbb{U}} \\ &\quad + L_1 \|u_1\|_{\mathbb{U}} |Z_{s_1-}^U - Z_{s_1-}^{g,U}| \\ &\leqslant L_1 \left(1 + \int_U \|u\|_{\mathbb{U}} \nu(\mathrm{d}u)\right) \int_0^{t_n} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}| \mathrm{d}r \\ &\quad + L_1 \|\Delta\chi_{\tau_1}\|_{\mathbb{U}} |Z_{\tau_1-}^U - Z_{s_1-}^U| + L_1(1 + |Z_{s_1-}^U|)\|\Delta\chi_{\tau_1} - u_1\|_{\mathbb{U}} \\ &\quad + L_1 \|u_1\|_{\mathbb{U}} \sup_{0\leqslant s\leqslant t_n} |Z_s^U - Z_s^{g,U}|. \end{aligned}$$

By **Assumption 1**, the definition of A_1, A_2 and the Gronwall inequality, we know that there exists a constant $C > 0$ such that

$$|Z_{s_1-}^U| < C, \quad |Z_{\tau_1-}^U - Z_{s_1-}^U| < C\varepsilon'.$$

Thus, on A_1 one have

$$\begin{aligned} \sup_{0\leqslant s\leqslant t_n} |Z_s^U - Z_s^{g,U}| &\leqslant \frac{L_1 \left(1 + \int_U \|u\|_{\mathbb{U}} \nu(\mathrm{d}u)\right)}{1 - L_1 \|u_1\|_{\mathbb{U}}} \int_0^{t_n} \sup_{0\leqslant s\leqslant r} |Z_s^U - Z_s^{g,U}| \mathrm{d}r \\ &\quad + \frac{L_1(\varepsilon' + \|u_1\|_{\mathbb{U}})C}{1 - L_1 \|u_1\|_{\mathbb{U}}} \varepsilon' + \frac{L_1(1 + C)}{1 - L_1 \|u_1\|_{\mathbb{U}}} \varepsilon'. \end{aligned}$$

The Gronwall inequality admits us to obtain that

$$\sup_{0\leqslant s\leqslant t_n} |Z_s^U - Z_s^{g,U}| \leqslant C\varepsilon'.$$

Taking $C\varepsilon' = \varepsilon/2$, one can attain that

$$\sup_{0\leqslant s\leqslant t_n} |Z_s^U - Z_s^{g,U}| < \varepsilon/2,$$

on $A_1 \cap A_2$. The proof is completed. $\square$

Next, we want to strengthen those conditions in **Assumptions 1 and 2** and give another support theorem.

Assumption 3.

(i) ξ and η are 3-times differentiable with bounded derivatives of all order between 1 and 3.

(ii) For any $u \in \mathbb{U}_0$, $\zeta(\cdot, u)$ is 3-times differentiable, and

$$\begin{aligned} \zeta(0,\cdot) &\in \bigcap_{2\leqslant q<\infty} L^q(\mathbb{U}_0,\nu) \\ \sup_x |\partial_x^r \zeta(x,\cdot)| &\in \bigcap_{2\leqslant q<\infty} L^q(\mathbb{U}_0,\nu), \quad 1 \leqslant r \leqslant 3, \end{aligned}$$

where $\partial_x^r \zeta(x,\cdot)$ stands for r order partial derivative of $\zeta(x,\cdot)$ with respect to x.

Under **Assumption 3**, by [5, Theorem 2-14, p.11], it holds that Eq.(1) has a unique solution which is still denoted by $Z_t(z_0)$ and the distribution of Z_t possesses a density. Our second main result of this section states as follows.

Theorem 2.5. *Suppose that* **Assumption 3** *is satisfied. Then* $supp(Z_t) = \mathbb{R}^d$ *for* $t \in [0,T]$.

Proof. By [5, Theorem 2-14, p.11], we know that the distribution of Z_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and clearly, the support of the probability density function is the whole $\mathbb{R}^d$, the support for the distribution of Z_t is also $\mathbb{R}^d$. The proof is completed. $\square$

3. Application

In the section, we will apply Theorem 2.2 to a problem on stochastic evolution equations with jumps on (separable) Hilbert spaces.

Let us begin with some notions and notations. Let $\mathbb{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. Let $L(\mathbb{H})$ be the set of all bounded linear operators $L:\mathbb{H}\to\mathbb{H}$ and $L_{HS}(\mathbb{H})$ be the collection of all Hilbert-Schmidt operator $L:\mathbb{H}\to\mathbb{H}$ equipped with the Hilbert-Schmidt norm $\|\cdot\|_{HS}$.

Let A be a linear, unbounded, negative definite and self-adjoint operator on $\mathbb{H}$ and $D(A)$ be the domain of the operator A. Let $\{e^{tA}\}_{t\geq 0}$ be the contraction C_0-semigroup generated by A. Let $L_A(\mathbb{H})$ be the collection of all densely defined closed linear operators $(L, D(L))$ on $\mathbb{H}$ so that $e^{tA}L$ can extend uniquely to a Hilbert-Schmidt operator still denoted by $e^{tA}L$ for any $t>0$. And then $L_A(\mathbb{H})$, endowed with the σ-algebra induced by $\{L \to \langle e^{tA}Lx, y\rangle_{\mathbb{H}} \mid t>0, x,y\in\mathbb{H}\}$, becomes a measurable space.

Let $\{\beta^i, i\in\mathbb{N}\}$ be a family of mutually independent one-dimensional Brownian motions on $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$. So, we can construct a cylindrical Brownian motion on $\mathbb{H}$ by

$$W_t := \sum_{i=1}^{\infty} \beta_t^i e_i, \quad \omega\in\Omega,\ t\in[0,\infty),$$

where $\{e_i, i \in \mathbb{N}\}$ is a complete orthonormal basis for $\mathbb{H}$ which will be specified later. It can be justified that the covariance operator of the cylindrical Brownian motion W is the identity operator I on $\mathbb{H}$. It is worthwhile to mention that W is not a process on $\mathbb{H}$. However, W can be realized as a continuous process on an enlarged Hilbert space $\tilde{\mathbb{H}}$, the completion of $\mathbb{H}$ under the inner product

$$\langle x, y\rangle_{\tilde{\mathbb{H}}} := \sum_{i=1}^{\infty} 2^{-i}\langle x, e_i\rangle\langle y, e_i\rangle, \quad x, y \in \mathbb{H}.$$

Next, we introduce a type of jump measures. Let $\lambda : \mathbb{U} \to (0,1)$ be a measurable function. Then, by Theorem I.8.1 of [8], we can construct an integer-valued random measure on $[0,\infty) \times \mathbb{U}$

$$N_\lambda : \mathscr{B}([0,\infty)) \times \mathscr{U} \times \Omega \to \mathbb{N}_0 := \mathbb{N} \cup \{0\} \cup \{\infty\}$$

with the predictable compensator $\lambda(u)\mathrm{d}t\nu(\mathrm{d}u)$:

$$\mathbb{E}N_\lambda(\mathrm{d}t, \mathrm{d}u, \cdot) = \lambda(u)\mathrm{d}t\nu(\mathrm{d}u).$$

Set

$$\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u) := N_\lambda(\mathrm{d}t, \mathrm{d}u) - \lambda(u)\mathrm{d}t\nu(\mathrm{d}u),$$

and then $\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u)$ is the associated compensated martingale measure of $N_\lambda(\mathrm{d}t, \mathrm{d}u)$. Moreover, we assume that $W_t, N_\lambda(\mathrm{d}t, \mathrm{d}u)$ are mutually independent.

Now consider the following stochastic evolution equation with jumps on $\mathbb{H}$

$$\begin{cases} dX_t = \{AX_t + b(X_t)\}dt + \sigma(X_t)dW_t + \displaystyle\int_{\mathbb{U}_0} f(X_{t-}, u)\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u), \quad 0 < t \leqslant T, \\ X_0 = x_0 \in \mathbb{H}, \end{cases} \tag{9}$$

where $b : \mathbb{H} \to \tilde{\mathbb{H}}$, $\sigma : \mathbb{H} \to L_A(\mathbb{H})$ and $f : \mathbb{H} \times \mathbb{U}_0 \mapsto \tilde{\mathbb{H}}$ are all Borel measurable mappings. Set $\|x\|_{\mathbb{H}} = \infty, x \notin \mathbb{H}$. For b, σ, f, we make the following assumption.

Assumption 4.

(i) There exists an integrable function $L_b : (0,T] \to (0,\infty)$ such that

$$\|e^{sA}(b(x) - b(y))\|_{\mathbb{H}}^2 \leqslant L_b(s)\|x - y\|_{\mathbb{H}}^2, \quad s \in (0,T], x, y \in \mathbb{H},$$

and

$$\int_0^T \|e^{sA}b(0)\|_{\mathbb{H}}^2 \mathrm{d}s < \infty.$$

(ii) There exists an integrable function $L_\sigma : (0,T] \to (0,\infty)$ such that for $\forall s \in (0,T]$ and $\forall x, y \in \mathbb{H}$

$$\|e^{sA}\left(\sigma(x) - \sigma(y)\right)\|_{HS}^2 \leqslant L_\sigma(s)\|x - y\|_{\mathbb{H}}^2$$

and

$$\int_0^T \|e^{sA}\sigma(0)\|_{HS}^2 \mathrm{d}s < \infty.$$

(iii) There exists an integrable function $L_f : [0,T] \to (0,\infty)$ such that

$$\|e^{sA}(f(x,u) - f(y,u))\|^2 \leqslant L_f(s)\|u\|_{\mathbb{U}}^2\|x - y\|_{\mathbb{H}}^2, \quad s \in [0,T], x, y \in \mathbb{H},$$
$$\|e^{sA}(f(x,u_1) - f(x,u_2))\|^2 \leqslant L_f(s)(1 + \|x\|_{\mathbb{H}})^2\|u_1 - u_2\|_{\mathbb{U}}^2, \quad u_1, u_2 \in \mathbb{U}_0,$$

and

$$\int_{\mathbb{U}_0} \|e^{sA}f(x,u)\|_{\mathbb{H}}^2\lambda(u)\nu(\mathrm{d}u) \leqslant L_f(s)(1+\|x\|_{\mathbb{H}})^2.$$

Under **Assumption 4**, [14, Theorem 3.2] admits us to obtain that Eq(9) has a unique mild solution, denoted by X_t.

Assumption 5.
The operator $-A$ has the following eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \dots$$

counting multiplicities.

The complete orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $\mathbb{H}$ is taken as the eigen-basis of $-A$ throughout the rest of the paper. Let $\mathbb{H}_n$ be the space spanned by $e_1, \cdots, e_n$ for $n \in \mathbb{N}$. Define

$$\begin{aligned}
&\pi_n : \mathbb{H} \to \mathbb{H}_n, \\
&\pi_n x := \sum_{i=1}^{n} \langle x, e_i\rangle_{\mathbb{H}} e_i, \quad x \in \mathbb{H},
\end{aligned}$$

and then π_n is the orthogonal project operator from $\mathbb{H}$ to $\mathbb{H}_n$.

Assumption 6.
For any $n \in \mathbb{N}, z \in \mathbb{H}_n$ and any open ball $B \subset \mathbb{H}_n$, there exists a point $u \in supp(\nu^{\mathbb{U}_0})$ such that $\pi_n f(z,u) \in B$.

In the following, we give out a support theorem under these assumptions.

Lemma 3.1. *Under* **Assumptions 4-6***, it holds that* $supp(X_t) = \mathbb{H}$ *for* $t \in [0,T]$.

Proof. Since $supp(X_t) \subset \mathbb{H}$, it is sufficiently only to show that $supp(X_t) \supset \mathbb{H}$. Furthermore, by Definition 2.1, we only need to prove that for any $h \in \mathbb{H}$ and $r > 0$,

$$\mathbb{P}\{\|X_t - h\|_{\mathbb{H}} < r\} > 0,$$

or equivalently,

$$\mathbb{P}\{\|X_t - h\|_{\mathbb{H}} \geqslant r\} < 1.$$

On one hand, put $A_n := A\mid_{\mathbb{H}_n}, b_n := \pi_n b$, $\sigma_n := \pi_n\sigma$ and $f_n := \pi_n f$. Thus, we obtain the following SDE with jumps in $\mathbb{H}_n$

$$\begin{cases} dX_t^n = \{A_n X_t^n + b_n(X_t^n)\}dt + \sigma_n(X_t^n)dW_t + \int_{\mathbb{U}_0} f_n(X_{t-}^n, u)\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u), \\ X^n(0) = \pi_n x_0. \end{cases} \tag{10}$$

It is easy to see that Eq.(10) is similar to Eq.(1). Moreover, it follows from **Assumptions 4-6** that the coefficients b_n, σ_n and f_n satisfy **Assumptions 1-2**. Thus, by Theorem 2.2, it holds that Eq.(10) has a unique solution which is denoted by X_t^n with $supp(X_t^n) = \mathbb{H}_n$. So, Definition 2.1 admits us to obtain that for any small $0 < \varepsilon < r$ and $0 < \eta < 1$

$$\mathbb{P}\{\|X_t^n - \pi_n x\|_{\mathbb{H}} \geqslant r - \varepsilon\} < 1 - \eta.$$

On the other hand, it follows from [14, Lemma 3.3] that

$$\lim_{n\to\infty} \mathbb{E}\|X_t^n - X_t\|_{\mathbb{H}}^2 = 0, \quad t \in [0,T].$$

Thus, by the Chebyshev inequality we know that there exists a $N \in \mathbb{N}$ such that for $n > N$,

$$\mathbb{P}\{\|X_t - X_t^n\|_{\mathbb{H}} \geqslant \varepsilon/2\} < \eta/2, \quad \mathbb{P}\{\|\pi_n x - x\|_{\mathbb{H}} \geqslant \varepsilon/2\} < \eta/2.$$

Finally, based on these inequalities, it holds that

$$\begin{aligned}
\mathbb{P}\{\|X_t - x\|_{\mathbb{H}} \geqslant r\} &\leqslant \mathbb{P}\{\|X_t - X_t^n\|_{\mathbb{H}} \geqslant \varepsilon/2\} + \mathbb{P}\{\|X_t^n - \pi_n x\|_{\mathbb{H}} \geqslant r - \varepsilon\} \\
&\quad + \mathbb{P}\{\|\pi_n x - x\|_{\mathbb{H}} \geqslant \varepsilon/2\} \\
&< \eta/2 + 1 - \eta + \eta/2 \\
&= 1.
\end{aligned}$$

So, the proof is completed. □

In order to present our main result in this section, we need to introduce the following assumption.

Assumption 7.

(i) There exists a Borel measurable mapping $\varrho : \mathbb{H} \to \mathbb{H}$ such that

$$b(x) = \sigma(x)\varrho(x),$$

(ii)

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \|\varrho(X_s)\|_{\mathbb{H}}^2 \mathrm{d}s + \int_0^T \int_{\mathbb{U}_0} \left(\frac{1-\lambda(u)}{\lambda(u)}\right)^2 \lambda(u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] < \infty.$$

Taking

$$\begin{aligned}
\Lambda_t : &= \exp\Big\{ -\int_0^t \langle \varrho(X_s), \mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}} - \frac{1}{2}\int_0^t \|\varrho(X_s)\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{U}_0} \log \lambda(u) N_\lambda(\mathrm{d}s, \mathrm{d}u) - \int_0^t \int_{\mathbb{U}_0} (1 - \lambda(u))\nu(\mathrm{d}u)\mathrm{d}s\Big\},
\end{aligned}$$

by [11, Theorem 6], we know that Λ_t is a exponential martingale under **Assumption 7. (ii)**. Define a new probability measure $\hat{\mathbb{P}}$ by

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \Lambda_T.$$

Thus, [14, Theorem 2.1] admits us to obtain that on the new filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \hat{\mathbb{P}})$, $\hat{W}_t := W_t + \int_0^t \varrho(X_s)\mathrm{d}s$ is a cylindrical Brownian motion, and the predictable compensator and the compensated martingale measure of $N_\lambda(\mathrm{d}t, \mathrm{d}u)$ are $\mathrm{d}t\nu(\mathrm{d}u)$ and $\hat{N}(\mathrm{d}t, \mathrm{d}u)$, respectively.

Now, we state the main result of this section.

Theorem 3.2. *Suppose that* **Assumptions 4-7** *are satisfied. Let $v : \mathbb{H} \to \mathbb{R}$ be a C^2 scalar function such that $[\nabla v(x)] \in D(A)$ for any $x \in \mathbb{H}$ and $\|A\nabla v(\cdot)\|_{\mathbb{H}}$ is bounded locally, and $\|A\nabla v(\cdot)\|_{\mathbb{H}} : \mathbb{H} \to [0, \infty)$ is continuous. Then the Girsanov density Λ_t for Eq.(9) has the following path-independent property:*

$$\Lambda_t = \exp\{v(x_0) - v(X_t)\}, \quad t \in [0, T],$$

if and only if

$$\begin{aligned} \varrho(x) &= (\sigma^*\nabla v)(x), & x\in\mathbb{H}, \quad (11)\\ \lambda(u) &= \exp\{v(x+f(x,u))-v(x)\}, & (x,u)\in\mathbb{H}\times\mathbb{U}_0, \quad (12)\end{aligned}$$

and v satisfies the following (infinite-dimensional) integro-differential equation,

$$\begin{aligned} &\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](x)+\frac{1}{2}\|\varrho(x)\|^2_{\mathbb{H}}+\langle x, A\nabla v(x)\rangle_{\mathbb{H}}\\ &+\int_{\mathbb{U}_0}\Big[e^{v(x+f(x,u))-v(x)}-1-\langle f(x,u),\nabla v(x)\rangle_{\tilde{\mathbb{H}}}e^{v(x+f(x,u))-v(x)}\Big]\nu(\mathrm{d}u)=0, \quad (13)\end{aligned}$$

where $\sigma^*(x)$ stands for the conjugate of $\sigma(x)$, ∇ and ∇^2 stand for the first and second Fréchet operators, respectively.

Although the proof of the above theorem is similar to [14, Theorem 4.3], we prove it here for the readers' convenience.

Proof. Firstly, let us prove the "only if" part. By the expression of Λ_t, it holds that

$$\begin{aligned} \log\Lambda_t &= -\int_0^t\langle\varrho(X_s),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}-\frac{1}{2}\int_0^t\|\varrho(X_s)\|^2_{\mathbb{H}}\,\mathrm{d}s-\int_0^t\int_{\mathbb{U}_0}\log\lambda(u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad-\int_0^t\int_{\mathbb{U}_0}\Big(1-\lambda(u)+\lambda(u)\log\lambda(u)\Big)\nu(\mathrm{d}u)\mathrm{d}s.\end{aligned}$$

Besides, [14, Proposition 1] admits us to obtain that

$$\begin{aligned} v(x_0)-v(X_t) &= -\int_0^t\langle AX_s,\nabla v(X_s)\rangle_{\mathbb{H}}\mathrm{d}s-\int_0^t\langle\varrho(X_s),\sigma^*\nabla v(X_s)\rangle_{\mathbb{H}}\mathrm{d}s-\int_0^t\langle(\sigma^*\nabla v)(X_s),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}\\ &\quad-\int_{\mathbb{U}_0}\Big[v(X_{s-}+f(X_{s-},u))-v(X_{s-})-\langle f(X_{s-},u),\nabla v(X_{s-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(u)\nu(\mathrm{d}u)\mathrm{d}s\\ &\quad-\int_{\mathbb{U}_0}\left[v(X_{s-}+f(X_{s-},u))-v(X_{s-})\right]\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad-\frac{1}{2}\int_0^t[Tr(\sigma\sigma^*)\nabla^2 v](X_s)\mathrm{d}s. \quad (14)\end{aligned}$$

Based on the uniqueness of decomposition for $\log\Lambda_t$, one can have that

$$\begin{aligned} &\varrho(X_s)=(\sigma^*\nabla v)(X_s), \quad \log\lambda(u)=v(X_{s-}+f(X_{s-},u))-v(X_{s-}),\\ &-\int_0^t\langle AX_s,\nabla v(X_s)\rangle_{\mathbb{H}}\mathrm{d}s-\frac{1}{2}\int_0^t\|\varrho(X_s)\|^2_{\mathbb{H}}\,\mathrm{d}s+\int_{\mathbb{U}_0}\langle f(X_{s-},u),\nabla v(X_{s-})\rangle_{\tilde{\mathbb{H}}}\lambda(u)\nu(\mathrm{d}u)\mathrm{d}s\\ &-\frac{1}{2}\int_0^t[Tr(\sigma\sigma^*)\nabla^2 v](X_s)\mathrm{d}s+\int_0^t\int_{\mathbb{U}_0}\Big(1-\lambda(u)\Big)\nu(\mathrm{d}u)\mathrm{d}s=0.\end{aligned}$$

Thus, by Lemma 3.1, it holds that (11)-(13) are right.

Next, we show "if" part. Combining (14) with (11)-(13), one can get that

$$\begin{aligned} v(x_0)-v(X_t) &= -\int_0^t\langle AX_s,\nabla v(X_s)\rangle_{\mathbb{H}}\mathrm{d}s-\int_0^t\langle\varrho(X_s),\sigma^*\nabla v(X_s)\rangle_{\mathbb{H}}\mathrm{d}s-\int_0^t\langle(\sigma^*\nabla v)(X_s),\mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}}\\ &\quad-\int_{\mathbb{U}_0}\Big[v(X_{s-}+f(X_{s-},u))-v(X_{s-})-\langle f(X_{s-},u),\nabla v(X_{s-})\rangle_{\tilde{\mathbb{H}}}\Big]\lambda(u)\nu(\mathrm{d}u)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
& -\int_{\mathbb{U}_0} \left[v(X_{s-}+f(X_{s-},u))-v(X_{s-})\right] \tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u) \\
& -\frac{1}{2}\int_0^t [Tr(\sigma\sigma^*)\nabla^2 v](X_s)\mathrm{d}s \\
= \ & -\int_0^t \langle \varrho(X_s), \mathrm{d}W_s\rangle_{\tilde{\mathbb{H}}} - \frac{1}{2}\int_0^t \|\varrho(X_s)\|^2_{\mathbb{H}}\,\mathrm{d}s - \int_0^t\int_{\mathbb{U}_0} \log\lambda(u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u) \\
& -\int_0^t\int_{\mathbb{U}_0}\Big(1-\lambda(u)+\lambda(u)\log\lambda(u)\Big)\nu(\mathrm{d}u)\mathrm{d}s \\
= \ & \log\Lambda_t.
\end{aligned}
$$

The proof is completed. □

Remark 3.3. *Comparing Theorem 3.2 with* [14, Theorem 4.3]*, one can find that, here* $\sigma(x)$ *may be degenerate or even could be zero.*

The above theorem gives a necessary and sufficient condition, and hence a characterization of path-independence for the density Λ_t of the Girsanov transformation for a stochastic evolution equation with jumps in terms of an infinite-dimensional integro-differential equation. Namely, we establish a bridge from Eq.(9) to an infinite-dimensional integro-differential equation.

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